RAN-1901001101010001

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M. A. (Mathematics) Part - I (External) Examination March - 2026

Mathematics Measure Theory

[Total Marks: 100

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) All questions are compulsory. (3) Follow usual notations and conventions. (4) Figures to the right indicate marks of the question.	Student's Signature

Q. 1. (a) Define outer measure of a set. Show that outer measure is translation invariant. Also if A is countable, show that $m^*(A) = 0$. **7**

(b) Let A be any set and $E_1, E_2, ..., E_n$ be a finite sequence of disjoint measurable sets. Then prove that $m^*(A \cap [\cup_{i=1}^{n} E_i] = \Sigma_{i=1}^{n} m^*(A \cap E_i)$. **7**

(c) If E_1 and E_2 are measurable sets then prove that $E_1 \cup E_2$ is measurable. **6**

OR

Q. 1. (a) Prove that the outer measure of an interval is its length. **7**

(b) Define measurable set. Prove that the family $\mathfrak{M}$ of measurable sets is an algebra. **7**

(c) Prove that the set E is measurable if and only if given $\in > 0$, there is an open set $0 \supset E$ with $m^*(0 \sim E) < \in$. **6**

Q. 2. (a) For an extended real valued function f defined on a measurable set, state all sufficient conditions for f to be measurable and prove their equivalence. **7**

(b) State and prove Lebesgue convergence theorem. **7**

(c) Let $\varphi = \Sigma_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \phi$ for $i \neq j$. If each E_i is a measurable set of finite measure, then show that $\int \varphi = \Sigma_{i=1}^{n} a_i m(E_i)$. **6**

OR

Q. 2. (a) Let f be a bounded function defined on a measurable set E of finite measure. Prove that f is measurable if and only if $\inf\limits_{f \le \psi} \int_E \varphi(x)dx$, $\sup\limits_{f \ge \psi} \int_E \varphi(x)dx$, for all simple functions φ and ψ. **7**

(b) State and prove Fatou's lemma. **7**

(c) Define characteristic function χ_A of a set A. Show that the function χ_A is measurable if and only if A is measurable. **6**

Q. 3. (a) Let f be a function of bounded variation on $[a, b]$, then prove that $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$. **7**

(b) If φ is convex on (a, b) and if x, y, x', y' are points of (a, b) with $x \le x' < y'$ and $x < y < y'$, then prove that $\frac{\varphi(y) - \varphi(x)}{y - x} \le \frac{\varphi(y') - \varphi(x')}{y' - x'}$. **7**

(c) If f is an absolutely continuous real valued function on $[a, b]$, then prove that f is of bounded variation on $[a, b]$, **6**

OR

Q. 3. (a) Define Sequence space l^∞ and prove that l^∞ is a complete metric space. **7**

(b) Prove that every finite dimensional subspace Y of a normed space X is complete. **7**

(c) Prove that the norms $\|x\|_\infty$ and $\|x\|_2$ defined on $\mathbb{R}^2$, as $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}}$ and $\|x\|_\infty = \{|\xi_1|, |\xi_2|\}$ for $x = (\xi_1, \xi_2)$, are equivalent.

Q. 4. (a) State and prove F. Riesz's lemma. **7**

(b) If a normed space X is finite dimensional, prove that every linear operator on X is bounded. **7**

(c) Let X and Y be vector spaces both real or complex. Let T: $\mathcal{D}(T) \to Y$ be a linear operator with domain $D(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. If dim $D(T) = n < \infty$, then prove that dim $\mathcal{R}(T) < n$. **6**

OR

Q. 4. (a) If a normed space X has the property that the closed unit ball $M = \{x \mid \|x\| \leq 1\}$ is compact, then prove that X is finite dimensional. **7**

(b) Let T: $\mathcal{D}(T) \to Y$ be a linear operator, where $D(T) \subset X$, X and Y are normed spaces. Prove that T is continuous if and only if T is bounded. **7**

(c) Prove that a linear functional f: $C[a, b] \to \mathbb{R}$ given by $f(x) = \int_a^b x(t)\, dt$, for all $x \in C[a, b]$ is bounded and $\|f\| = b - a$. **6**

Q. 5. (a) Define Dual space. Show that the dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$. **7**

(b) State and prove Vector Minimizing Theorem. **7**

(c) Show that the space l^P with $p \neq 2$ is not a Hilbert space. **6**

OR

Q. 5. (a) Prove that a finite dimensional vector space is algebraically reflexive. **7**

(b) In an inner product space, show that the norm satisfy: **7**

(i) $||\langle x,y\rangle| \leq \|x\|\,\|y\|$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

(c) If Y is a closed subspace of a Hilbert space H, then prove that $Y = Y^{\perp\perp}$. **6**